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Asset Price Bubbles with Low Interest Rates: Not All Bubbles are Alike

Jacopo Bonchi*

Abstract

I extend a standard two-period OLG model to investigate the interplay between the risks of a binding zero lower bound and asset price bubbles in a low interest rates environment. The nature of the bubble is crucial when the risk-free real interest rate is low because there is a negative natural interest rate. Bubbles are *fully* leveraged when they are sustained by borrowers, or they are *fully* unleveraged when they are sustained by lenders. Leveraged bubbles emerge naturally when there is a negative natural interest rate, and they are more likely to collapse. Unleveraged bubbles appear, in contrast, if the natural rate of interest is extremely low and the probability of the bubble bursting is not extremely high. Both bubbles are more likely to emerge with a high inflation target and will potentially be larger, but only leveraged bubbles substantially mitigate the risk of a zero lower bound episode by raising the natural rate of interest.

JEL Codes: E43, E44, E52

Keywords: zero lower bound, low interest rates, asset price bubbles, inflation target

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Non-technical summary

Persistently low interest rates, caused by a negative natural rate of interest, expose the economy to two risks above all: zero lower bound (ZLB) episodes become more frequent and rational asset prices bubbles are more likely to emerge.

I investigate in this paper the possible interaction between these two risks by distinguishing leveraged bubbles from unleveraged bubbles. The existence of rational bubbles is partly dictated by the inflation target, whose level also affects the risk of a liquidity trap. Equally bubbles can temporarily raise the natural rate of interest, giving the central bank more space to cut the nominal rate in a recession. However, not all bubbles are alike. If they serve as a store of value without fostering credit growth in the form of unleveraged bubbles, the bubble bursting does not necessarily provoke a financial crisis. Leveraged bubbles in contrast foster a credit boom that can trigger a financial crisis when they burst.

The theoretical model I develop (an extended two-period OLG model) shows the two bubble types emerge under different conditions at low interest rates. Unleveraged bubbles occur only at an extremely low natural interest rate and with a low probability of bursting, while leveraged bubbles always occur at a negative natural interest rate and the probability they will burst is high. Leveraged bubbles are also accompanied by an increase in the natural rate of interest which strongly reduces the possibility of a ZLB episode. A similar effect comes from unleveraged bubbles too, but these do not necessarily appear and they bring a smaller increase in the natural rate.

A higher inflation target, though it mitigates the risk of falling into a liquidity trap, fosters asset price bubbles of any type, because both unleveraged and leveraged bubbles are more likely to occur and they will potentially be larger.

Contents

"However, not all asset price bubbles are alike.... In particular, some asset price bubbles can have more significant economic effects, and thus raise additional concerns for economic policymakers, by contributing to financial instability"

-Frederic Mishkin, Financial Stability Review no.12 2008, Banque de France

1 Introduction

In this work I investigate the source and effect of leveraged and unleveraged bubbles in order to shed light on the risks and gains of different bubble types when interest rates are persistently low.

Low interest rates have shaped the global economy in 2008–2018. The historical fall in the risk-free interest rates in the advanced economies is well and widely documented in the empirical literature, which relates it to the decline in the natural interest rate that is consistent with output at the potential level (Rachel and Smith, 2015; Laubach and Williams, 2016; Holston et al., 2017). The drivers of this decline are slow-moving forces whose pattern has not been reversed during the Great Recession and the ensuing recovery (IMF, 2014).[1] This explains the persistence of low interest rates, which exposes the economy to two risks above all.

The first is that zero lower bound (ZLB) episodes become more frequent. If a negative shock hits the economy, the central bank will not have enough space to lower its policy rate and the resulting recession will be deeper and longer (Williams, 2009; Kiley and Roberts, 2017). The second risk is that financial instability is heightened, and this play out in several forms. Risk-taking behaviours and borrowing are encouraged because only risky investments are profitable and credit is cheap, while if the real interest rate falls below the economy's growth rate, the consequent leap in asset prices is rational (Baldwin and Teulings, 2014).

I aim to investigate the possible interaction between these two risks by focusing on *rational* asset price bubbles as a form of financial instability. Their existence is partly dictated by the inflation target, whose level can also exacerbate or mitigate the risk of a liquidity trap, through a no-arbitrage condition that links the real interest rate and the growth rate of the economy (Samuelson, 1958; Tirole, 1985).[2] Equally bubbles can temporarily raise the natural rate of interest by serving as a store of value and collateral, and so they give the central bank more room to cut the policy rate in a recession (Bonchi, 2017).[3] The extent to which this gain is tied to potential output losses from a bubble bursting crucially depends on the bubble type. Asset price bubbles are not inherently harmful. If they serve as a store of value without fostering credit growth in the form of unleveraged bubbles, the economic cost of a bubble bursting is limited and it does not necessarily provoke a financial crisis. Leveraged bubbles in contrast foster a credit boom that can painfully hurt the economy, and they are more likely to trigger a financial crisis

[1] For a complete list of these forces see Summers (2014, 2015) and Baldwin and Teulings (2014). For a quantitative estimation of their relative importance see, among others, Eggertsson et al. (2017).

[2] Although this condition was first derived within OLG models without prices, the inflation level affects the real interest rate through the Fisher equation in a monetary economy.

[3] A similar idea is contained in Asriyan et al. (2016), where the collapse of a bubble drags the economy into a liquidity trap by destroying a large portion of the total collateral.

(Jordá et al., 2015). To understand what bubble emerges for different inflation targets in a low rates environment and what is most effective at raising the natural interest rate, I develop a two-period OLG model with income inequality, downwardly rigid nominal wages and rational bubbles.

Income inequality shapes the characteristics of young households in the credit market, as low-income households are borrowers and high-income ones are lenders. When income is very concentrated among richer households, the natural rate of interest turns negative.[4] If the inflation target is too low, the central bank cannot drive the real interest rate to its natural level using standard monetary policy tools, and then the economy gets stuck in a low interest rate equilibrium, which features a binding ZLB and a long-lasting recession because of nominal wage rigidities. In contrast, the economy does not experience a low interest rate at a sufficiently high inflation target, but rational bubbles can still emerge anyway. Bubbles are *fully* unleveraged, when lenders are the only owners of bubbly assets, while they are *fully* leveraged, when only borrowers purchase them. A higher inflation target fosters asset price bubbles of any type, because both unleveraged and leveraged bubbles are more likely to occur and they will potentially be larger.

However, the condition for the existence of the two bubble types is not the same, and nor is the probability of them bursting. Unleveraged bubbly episodes occur only at an extremely low natural interest rate and they have a low probability of bursting, which makes them less risky. Leveraged bubbly episodes always occur at a negative natural interest rate, and the probability they will burst is high, because borrowing happens through a defaultable debt contract which induces risk-shifting.

The heightened risk of a leveraged bubble is compensated by its positive effect on the natural interest rate. In fact, leveraged bubbles are accompanied by an increase in the natural rate of interest, which strongly reduces the risk of a ZLB episode. A similar effect comes from unleveraged bubbles too, but these do not necessarily appear and they bring a smaller increase in the natural rate.

This paper is inspired by the literature on low interest rates (Blanchard et al., 2010; Stiglitz, 2012; Ball, 2014; Summers 2014, 2015; Gordon, 2015; Rogoff, 2016; Eggertsson et al., 2016, 2017). Stiglitz (2012) argues that increasing income inequality puts downward pressure on interest rates and in this way provides fertile ground for bubbles. The idea that inequality depresses the natural interest rate is formalised by Eggertsson et al. (2017), who develop a tractable OLG model to represent the main sources of low interest rates, but do not investigate the effect of asset price bubbles. Furthermore, I borrow from Blanchard et al. (2010) and Ball (2014) the idea that a higher inflation target relaxes the ZLB constraint, and I study how different inflation targets affect the existence of leveraged and unleveraged bubbles.

My work also relates to the literature on rational asset price bubbles, which includes, among others, Samuelson (1958), Tirole (1985), Weil (1987), Martin and Ventura (2011, 2012), Galí (2014), Asriyan et al. (2016), and Bengui and Phan (2016, 2018). This literature originates within the OLG framework, but rational bubbles have recently been

[4] According to Eggertsson et al. (2017), the decline in the natural rate is mainly caused by demographic and technological factors, not by income inequality. I do not aim to replicate precisely the main sources of low interest rates, rather I use income inequality to determine them and, above all, to distinguish between the two bubble types through the identity of their owners.

analysed in infinite horizon models too (e.g., Kocherlakota, 2008; Hirano and Yanagawa, 2016; Dong et al., 2017; Miao and Wang, 2018; and Kiyotaki and Moore, 2018). I introduce leveraged and unleveraged bubbles in an OLG model, along the lines of Bengui and Phan (2016, 2018). Their analysis is limited to an endowment economy and I extend it to a production economy with non-neutral monetary policy. I also add income inequality to create a low interest rates environment and downwardly rigid wages to make it highly persistent. Unlike in Galí (2014), monetary policy can influence the existence and the size of a rational bubble in my model. Raising the inflation target fosters bubbles and this does not go through the net worth of entrepreneurs like in Dong et al. (2017), but rather bubbly assets emerge because a higher inflation level pushes the real interest rate down.

The rest of the paper is organised as follows. In sections 2 and 3 I present the model and illustrate the steady state equilibrium in a bubbleless economy, and with unleveraged and leveraged bubbles. Section 4 concludes.

2 Model

I study a two-period OLG economy without capital and with zero population growth. Agents in this model form expectations rationally and the size of generations is normalised to 1. I extend the standard OLG framework in three directions.

First, I assume income inequality and the existence of a credit market in which young households borrow and lend. There are two agents, as depicted in Figure 1. χ young households, which are referred to as *lenders*, receive a high income and save for retirement. The remaining ones are *borrowers*. They receive a low income and borrow by issuing a one-period bond, but they cannot smooth consumption over time because of a debt constraint. Borrowers also pay a lump-sum tax (T) to finance social security benefits when they are old.[5] The equilibrium condition for the credit market is:

$$\chi d_t^L = (1-\chi)\, d_t^B \tag{1}$$

where d_t^B and d_t^L are respectively the amount of funds demanded by each borrower and supplied by each lender.

The credit market is incomplete, because borrowers cannot commit to paying all their outstanding debt. They issue a non-contingent standard debt contract, which is defaultable.[6] If there is a default, lenders can repossess an amount $D \in (0, T)$ that consists of the fundamental collateral, and a bubbly collateral $\phi p_{t+1}^b b_t^B$, where the parameter $\phi \in [0, 1]$ measures the pledgeability of the bubbly assets (Bengui and Phan, 2018). I also assume the gross real interest rate $(1+r_t)$ that is charged on the borrower's debt does not depend on the size of the loan.[7]

[5] In principle, high-income households could also borrow and default, but if a fraction of their savings can be seized, they will never default and the optimal level of borrowing will be zero. This result is shown in a similar setting by Bengui and Phan (2018).

[6] A microfoundation for this contract is given in Ikeda and Phan (2016).

[7] I do not assume "credit pooling" like Bengui and Phan (2018), though it provides a more accurate representation of a leveraged bubble. This assumption would make the model less tractable without giving any additional insight, so I prefer to stick to the assumption of Allen and Gale (2000) and Ikeda and Phan (2016).

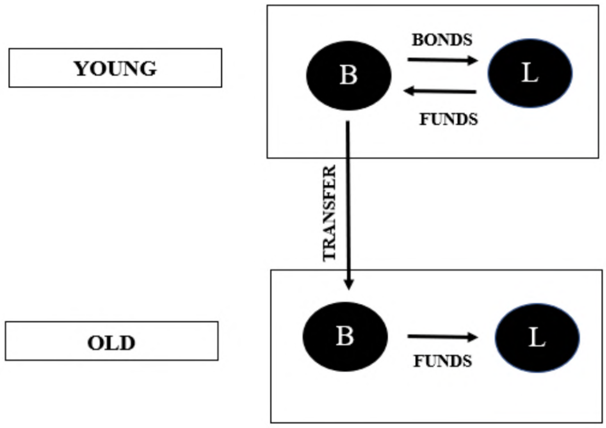

Figure 1: Structure of my extended OLG model

Second, workers are unwilling to accept a nominal wage below a minimum level. Downward wage rigidity allows for the non-neutrality of monetary policy, which is specified in terms of a standard Taylor rule.[8]

Third, under certain conditions rational asset price bubbles can emerge. A bubbly asset has a fundamental value of zero, but it is purchased at a positive price when the buyer expects to sell it at a higher price. The value of the bubble is $\theta_t p_t^b$; p_t^b is its price, if it does not collapse; $\{\theta_t\}_{t=0}^{\infty}$ is a binary random variable that captures the events of the bubble crashing ($\theta_t = 0$) or surviving ($\theta_t = 1$). There is a fixed probability ρ of the bubble collapsing, and if the bubble has already crashed, it never re-emerges. The bubbly asset has fixed unit supply, so the market for bubbles clears if:

$$\chi b_t^L + (1-\chi)\, b_t^B = 1 \tag{2}$$

where b_t^L and b_t^B are the bubble purchases of lenders and borrowers.

In this section, I first explain the source of income inequality and the downward wage rigidity within a broader analysis of the supply-side of the model. Then I will analyse the behaviour of the central bank and study the maximisation problem of the two households.

2.1 Firms

Only young households run firms and supply labour. The production function of firms is:

$$Y_t = L_t^{\alpha} \tag{3}$$

[8] This assumption does not represent the standard way to introduce a non-neutral monetary policy, but it is crucial to make low interest rates persistent, as mentioned above. Furthermore, evidence of downward nominal wage rigidity during the US Great Recession is provided by Fallick, Lettau and Wascher (2016), while Schmitt-Grohé and Uribe (2016) document downwardly rigid wages in emerging countries.

where $0 < \alpha < 1$. As goods and labour markets are perfectly competitive,[9] firms take the price of goods (P_t) and labour services (W_t) as given and maximise profits:

$$Z_t = P_t Y_t - W_t L_t \tag{4}$$

subject to (3). The optimality condition for this problem is the labour demand:

$$\frac{W_t}{P_t} = \alpha L_t^{\alpha-1} \tag{5}$$

Aggregate labour demand, L_t, is the weighted average of the demand for labour services from borrowers (L_t^B) and lenders (L_t^L):

$$L_t = (1-\chi) L_t^B + \chi L_t^L \tag{6}$$

Young agents are equally skilled and they supply labour inelastically, so the aggregate labour supply coincides with the labour endowment of the economy $\bar{L} = (1-\chi)\bar{L}^B + \chi \bar{L}^L$. Lenders have a higher labour endowment than borrowers, and the demand for the labour services of borrowers, and those of lenders, is a constant share of the aggregate labour demand, which is equal to the corresponding share of the total labour endowment:[10]

$$(1-\chi)\frac{L_t^B}{L_t} = (1-\chi)\frac{\bar{L}^B}{\bar{L}} = \epsilon \tag{7}$$

The total income of borrowers and that of lenders are:

$$Y_t^B = \frac{Z_t}{P_t} + \frac{W_t}{P_t} L_t^B$$
$$Y_t^L = \frac{Z_t}{P_t} + \frac{W_t}{P_t} L_t^L$$

$Y_t^L > Y_t^B$ directly follows from (7) and $\bar{L}^L > \bar{L}^B$.

Workers do not accept a wage lower than a minimum level (Schmitt-Grohé and Uribe, 2016). The nominal wage is accordingly downwardly rigid:

$$W_t = \max\left(\overline{\Pi} W_{t-1}, P_t \alpha \bar{L}^{\alpha-1}\right) \tag{8}$$

$\overline{\Pi} W_{t-1}$ is the lower bound on the nominal wage, where $\overline{\Pi} > 1$ is the gross inflation target,[11] and $P_t \alpha \bar{L}^{\alpha-1}$ is the flexible nominal wage that clears the labour market. If market clearing requires an increase in W_t from the previous period of more than the inflation target, the nominal wage equals its flexible level and the labour market clears ($L_t = \bar{L}$). However, if instead an increase of less than the inflation target is necessary to clear the labour market, $W_t = \overline{\Pi} W_{t-1}$ and involuntary unemployment arises ($L_t < \bar{L}$).

[9] The downward wage rigidity does not alter the structure of the labour market, because workers and employers are wage takers anyway.

[10] This implies that any fall in labour demand causes a proportional decline in the demand for the labour services of borrowers and lenders. This means the low interest rate equilibrium analysed below does not redistribute resources among young households.

[11] Unlike Schmitt-Grohé and Uribe (2016), I do not assume a constant degree of wage rigidity, which is in contrast measured by the inflation target. This allows me to generate a low interest rate equilibrium characterised by a positive inflation level.

2.2 The Central Bank

The central bank sets the gross nominal interest rate according to a standard Taylor rule:

$$1 + i_t = \max\left(1, \left(1 + r^f\right)\overline{\Pi}\left(\frac{\Pi_t}{\overline{\Pi}}\right)^{\phi_\pi}\right) \tag{9}$$

where $\phi_\pi > 1$, a property which is referred to as the *Taylor principle.* $\left(1 + r^f\right)\overline{\Pi}$ is the central bank target for the gross nominal interest rate and r^f is the natural rate of interest. The central bank stabilises inflation around the targeted level $\overline{\Pi}$ as long as the nominal interest rate does not hit the ZLB $(1 + i_t > 1)$. The standard Fisher equation holds:

$$1 + r_t = (1 + i_t)\, E_t \Pi_{t+1}^{-1} \tag{10}$$

where $\Pi_{t+1} = \frac{P_{t+1}}{P_t}$ is the gross inflation rate at $t + 1$.

Equations (9) and (10) play a crucial role in the analysis that follows. As shown in equation (9), it is less likely for the central bank to hit the ZLB at a high inflation target, even if the natural interest rate is negative $(1 + r^f < 1)$. On the other hand, high inflation puts downward pressure on the real interest rate through inflation expectations in equation (10), fostering rational bubbles. Asset price bubbles that emerge at a low interest rate mitigate the risk of a ZLB episode by raising r^f, as I will discuss below.

2.3 Households

I assume borrowers and lenders have logarithmic preferences, and denote consumption at young and old ages with $C_{y,t}^i$ and $C_{o,t+1}^i$, where $i \in \{B, L\}$.

At young age, borrowers get income Y_t^B, which is insufficient to smooth consumption over time. They choose the optimal amount of borrowing and bubble holdings by solving the maximisation problem:[12]

$$\max_{d_t^B, b_t^B} E_t\left(\ln C_{y,t}^B + \beta \ln C_{o,t+1}^B\right)$$

subject to

$$C_{y,t}^B = Y_t^B + d_t^B - \theta_t p_t^b b_t^B - T \tag{11}$$

$$C_{o,t+1}^B = T + \theta_{t+1} p_{t+1}^b b_t^B - (1 - \gamma_{t+1})(1 + r_t)\, d_t^B - \gamma_{t+1}\left(D + \phi\theta_{t+1} p_{t+1}^b b_t^B\right) \tag{12}$$

$$b_t^B \geq 0 \tag{13}$$

$$(1 + r_t)\, d_t^B \leq D + \phi p_{t+1}^b b_t^B \tag{14}$$

[12] I impose:

$$D < \frac{T}{1+\beta} - \frac{\beta}{1+\beta}(1 + r_t)\left(Y_t^B - T\right)$$

and so borrowers in a bubbleless economy cannot take on enough debt to implement the optimal consumption plan.

Borrowers optimally choose to repay or default at time $t+1$, according to the rule:

$$\gamma_{t+1} = 1\left\{(1+r_t)\, d_t^B > D + \phi\theta_{t+1}p_{t+1}^b b_t^B\right\} \tag{15}$$

1 is an indicator function, which takes the value 1 or 0. If repaying is more costly than defaulting, borrowers go bankrupt as shown in (15). If $\gamma_{t+1} = 0$, they repay all their outstanding debt, because defaulting is the most expensive option. Imposing the credit constraint (14) means that borrowers default only if the bubble bursts. The first order conditions for this problem are:

$$\lambda_{b,t}^B = \frac{1}{C_{y,t}^B} p_t^b - \beta E_t \frac{\left[(1-\gamma_{t+1}\phi)\,\theta_{t+1}\right] p_{t+1}^b}{C_{o,t+1}^B} - \lambda_{d,t}^B \phi p_{t+1}^b$$

$$\lambda_{d,t}^B (1+r_t) = \frac{1}{C_{y,t}^B} - \beta E_t \left[\frac{(1-\gamma_{t+1})(1+r_t)}{C_{o,t+1}^B}\right]$$

where $\lambda_{b,t}^B$ and $\lambda_{d,t}^B$ represent the shadow cost of the constraints (13) and (14). Substituting for $\lambda_{d,t}^B$ in the first condition gives:

$$\lambda_{b,t}^B = \frac{1}{C_{y,t}^B}\left[p_t^b - \frac{\phi p_{t+1}^b}{(1+r_t)}\right] - \beta(1-\rho)(1-\phi)\left(\frac{1}{C_{o,t+1}^B}\right) p_{t+1}^b \tag{16}$$

The first term on the right-hand side is the marginal cost of bubbles net of their collateral value, while the second term, which includes their discounted expected return, is the marginal benefit. If the cost is equal to the benefit, $\lambda_{b,t}^B$ is 0 and the demand for bubbles is positive; if the cost is higher than the benefit though, $\lambda_{b,t}^B > 0$ and the optimal bubble holding is 0.

Lenders get a sufficiently high income Y_t^L to save and implement the optimal consumption plan. Their maximisation problem is:

$$\max_{d_t^L, b_t^L} E_t \left(\ln C_{y,t}^L + \beta \ln C_{o,t+1}^L\right)$$

subject to:

$$C_{y,t}^L = Y_t^L - d_t^L - \theta_t p_t^b b_t^L \tag{17}$$

$$C_{o,t+1}^L = \theta_{t+1} p_{t+1}^b b_t^L + (1-h_{t+1})(1+r_t)\, d_t^L \tag{18}$$

$$b_t^L \geq 0 \tag{19}$$

In the event of default, lenders can repossess only a share of their original claims. The fraction of losses on loans is the haircut h_{t+1}, which is a random variable:

$$h_{t+1} = \begin{cases} 0 & \gamma_{t+1} = 0 \\ 1 - \frac{(1-\chi)}{\chi}\frac{D}{(1+r_t)d_t^L} & \gamma_{t+1} = 1 \quad (default) \end{cases} \tag{20}$$

If there is no default, the haircut is zero. This happens when low-income households do not borrow against the bubbly collateral, or they do, but the bubbles do not burst. If borrowers pledge bubbly assets and their value collapses to zero ($\theta_t = 0$), they default according to (15). The aggregate fundamental collateral $(1-\chi)D$, which is a fraction of the total claims $\chi(1+r_t)\,d_t^L$, is distributed evenly to lenders, and the remaining fraction of the outstanding debt represents the haircut on loans. The optimality conditions of the lenders' problem are:

$$\frac{1}{C_{y,t}^L} = \beta E_t\left[\frac{(1-h_{t+1})(1+r_t)}{C_{o,t+1}^L}\right] \tag{21}$$

$$\lambda_{b,t}^L = \frac{1}{C_{y,t}^L}p_t^b - \beta E_t\left[\frac{\theta_{t+1}p_{t+1}^b}{C_{o,t+1}^L}\right]$$

$\lambda_{b,t}^L$ denotes the lagrange multipliers associated with the constraint on lenders (19). The first condition is the Euler equation, where the return from bonds also includes the expected value of the haircut, while the second condition defines the optimal choice of bubbles. As lenders do not borrow, they do not use bubbles as collateral and they only benefit from the discounted expected return $\beta E_t\left[\theta_{t+1}p_{t+1}^b\right]$. Combining the two conditions yields:

$$\lambda_{b,t}^L = \beta E_t\left(\frac{1}{C_{o,t+1}^L}\right)p_t^b\left[E_t\left(1-h_{t+1}\right)(1+r_t) - (1-\rho)\left(\frac{p_{t+1}^b}{p_t^b}\right)\right] \tag{22}$$

The demand from lenders for bubbles is positive ($\lambda_{b,t}^L = 0$) only if the expected growth rate of the bubble price is equal to the expected return from bonds.

The no-arbitrage condition (22), along with (16), stresses the different reasons why lenders and borrowers hold bubbles. Lenders need an alternative store of value when there are few investment opportunities, while borrowers hold bubbly assets because of their collateral value, which crucially depends on ϕ. When bubbles are highly pledgeable, a high percentage of their value turns into credit and borrowers buy them to collect extra funds. As credit is fostered in this case, bubbles are leveraged if they are partially or fully held by borrowers, and they are unleveraged if lenders buy at least a fraction of the bubbly assets.

3 Equilibrium

Given W_{-1}, d_{-1}^L, $p_0^b \geq 0$ and θ_0, a competitive equilibrium consists of the prices $\left\{P_t, W_t, r_t, i_t, p_t^b\right\}$, the quantities $\left\{d_t^L, b_t^L, d_t^B, b_t^B, C_{y,t}^L, C_{o,t}^L, C_{y,t}^B, C_{o,t}^B, Y_t, Z_t, L_t, L_t^L, L_t^B\right\}$, the default decision γ_{t+1} and the haircut h_{t+1} that solve (1), (2), (3), (4), (5), (6), (7), (8), (9), (10), (11), (12), (14), (15), (16), (17), (18), (20), (21) and (22). This equilibrium is bubbleless for $p_0^b = 0$, while it is bubbly for $p_0^b > 0$. In the rest of this section, I will focus on the bubbleless and bubbly steady state equilibria, where the variables take a constant value. First I will study a bubbleless steady state, which can be one with full employment and a positive nominal interest rate or one with a persistent recession and binding ZLB, depending on the level of the inflation target. Then I will investigate how unleveraged and leveraged bubbly equilibria arise and what their features are.

I restrict the analysis of the bubbly equilibrium to the cases of *fully* unleveraged and *fully* leveraged bubbles, in order to study the two bubble types in isolation. A *fully* unleveraged bubble can arise for $\phi = 0$. As bubbly assets cannot be collateralised, borrowers have no incentive to hold them, and only lenders invest in bubbles if this is more profitable than buying bonds. In contrast, a precondition for the existence of *fully* leveraged bubbles is $\phi = 1$. If bubbles can be fully pledged in the credit market, borrowers buy them at a sufficiently low real interest rate. Furthermore, the possibility of default partially shifts the downside risk of the bubbly investment to lenders, inducing borrowers to hold even highly risky bubbles. This *risk-shifting* raises the real interest rate compared to that in a bubbleless economy, and so lenders invest all their savings in bonds, which guarantee a higher return than bubbly assets.[13] A formal proof that bubbles are fully unleveraged or leveraged for these two calibrations of the parameter ϕ is given in Appendix A.

3.1 A Bubbleless Economy

Before analysing the steady state of the model, I have to define the real interest rate that clears the credit market. This market will play a crucial role in the following arguments, because it transmits the effect of bubbles to the economy. Low-income households are borrowing-constrained ($d_t^B = D/1 + r_t$) and $h_{t+1} = 0$ in a bubbleless economy. The credit market clearing condition (1) can be rearranged as follows:

$$d_t^L = \frac{(1-\chi)}{\chi} d_t^B$$

I express the demand for credit, the right-hand side of this equation, as D_t^c and the supply of credit on the left-hand side as S_t^c. The credit demand, taken at the steady state, can be written as:

$$D^c = \frac{(1-\chi)}{\chi} \frac{D}{(1+r)} \tag{23}$$

Combining (17), (18) and (21) yields the supply of credit:

$$S^c = \frac{\beta}{1+\beta} Y^L \tag{24}$$

The market for credit clears at the equilibrium real interest rate:

$$(1+r) = \frac{(1-\chi)}{\chi} \frac{(1+\beta)\, D}{\beta Y^L} \tag{25}$$

which is obtained by equating (23) and (24). Although the income of lenders is endogenously determined by output, χY^L is a constant share of Y because of assumption (7), and so lenders having a large share of the total income results in a low equilibrium real

[13] More precisely, there are two threshold values for ϕ, like in Bengui and Phan (2016). Values below the lower threshold correspond to fully unleveraged bubbles, while values above the higher threshold are associated with fully leveraged bubbles. As ϕ is below the lower threshold or above the higher threshold in the cases analysed, and *mixed* leveraged and unleveraged bubbles ($0 < \phi < 1$) are not considered, I do not provide a formal characterisation of the two thresholds, which is left for future research.

interest rate. This is the source of a negative natural rate of interest, which is the market clearing real interest rate at the full employment level of production:

$$\left(1+r^{f}\right)=\frac{(1-\chi)}{\chi}\frac{(1+\beta)\,D}{\beta Y^{f,L}} \tag{26}$$

where $\chi Y^{f,L}$ is the fraction of the potential output attributed to lenders.

The steady state can be expressed by aggregate supply and demand, which are both characterised by two regimes like in Eggertsson et al. (2017). The regime of supply depends on the downward wage rigidity (8). For $\Pi \geq \overline{\Pi}$, aggregate supply (AS) can be computed from (3), (5) and (8):

$$Y_{AS}=\bar{L}^{\alpha}=Y^{f} \tag{27}$$

By combining the same equations, AS becomes:

$$Y_{AS}=\left(\frac{\Pi}{\overline{\Pi}}\right)^{\frac{\alpha}{1-\alpha}}Y^{f} \tag{28}$$

for $\Pi < \overline{\Pi}$. If the gross inflation rate is equal to or higher than the target, wages are flexible, so the aggregate labour demand equals the economy's labour endowment $\bar{L}$ and output is at the potential level. If the inflation rate is lower than the target, the wage cannot equate its market clearing level, which falls below the lower bound in (8), and involuntary unemployment arises, leaving output at a level below its potential. The resulting positive relation between inflation and output is a consequence of the real wage being too high; as inflation rises, the real wage falls, stimulating labour demand and output.[14] Equation (27) is represented by a vertical segment in Figure 3.1, while equation (28) is depicted as an upward sloping curve. The kink point at which the AS curve turns to become upward sloping corresponds to $\Pi=\overline{\Pi}$.

The regime of aggregate demand (AD) is determined by equation (9). When the nominal interest rate is positive (i_t>0), the following AD can be derived from the equations (9), (10) and (25):

$$Y_{AD}=(1-\chi)\,Y^{B}+(1-\chi)\left(\frac{1+\beta}{\beta}\right)\frac{\kappa}{\Pi^{\phi_{\pi}-1}}D \tag{29}$$

where $\kappa=\overline{\Pi}^{\phi_{\pi}-1}\left(1+r^{f}\right)^{-1}$. Combining the same equations yields a different AD with a binding ZLB (i_t=0):

$$Y_{AD}=(1-\chi)\,Y^{B}+(1-\chi)\left(\frac{1+\beta}{\beta}\right)\Pi D \tag{30}$$

Equation (29) expresses a negative relation between inflation and output, which is plotted as a downward sloping curve in Figure 2. This relation turns positive in a liquidity trap, as shown by the upward sloping segment of the AD curve in the same figure. At

[14] Equation (28) incorporates the idea of a long-run trade-off between inflation and output at low inflation levels. This idea was first put forward by Tobin (1972), and then formalised by Benigno and Ricci (2011) and Daly and Hobijn (2014), among others.

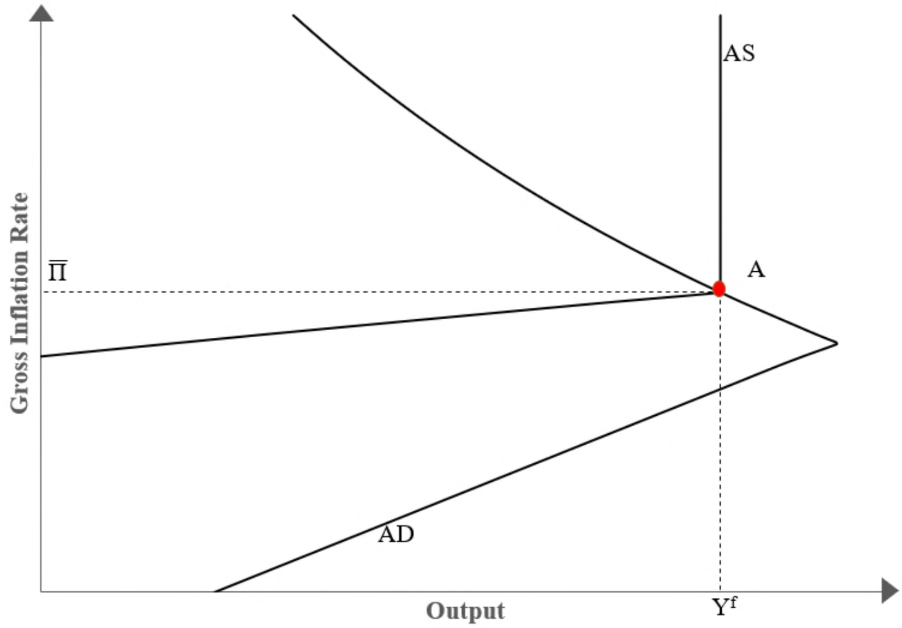

Figure 2: Aggregate Demand and Supply in a Bubbleless Economy

greater distances from the ZLB, the central bank reacts to higher inflation by raising the policy rate more than proportionally ($\phi_\pi > 1$), which contracts demand and so stabilises inflation around the targeted level. In a liquidity trap, ordinary monetary policy tools do not allow the central bank to equate the real interest rate to its natural level. Therefore the real interest rate is determined exclusively by the inflation rate in equation (10), and when inflation rises, the real rate falls and demand increases. The inflation level at which the ZLB becomes binding is represented graphically by a kink in the AD curve (Figure 2) and denoted by Π_{kink}. It is obtained by equating the two arguments in the max operator of (9) and solving for the inflation rate:

$$\Pi_{kink} = \left(\frac{1}{1+r^f}\right)^{\frac{1}{\phi_\pi}} \overline{\Pi}^{\frac{\phi_\pi - 1}{\phi_\pi}}$$

The relation between Π_{kink} and $\overline{\Pi}$ is crucial for determining the steady state equilibrium at a negative natural interest rate. When $(1 + r^f) < 1$ because of income inequality, $\Pi_{kink} \leq \overline{\Pi}$ implies:

$$(1 + r^f) \geq \frac{1}{\overline{\Pi}} \tag{31}$$

The inflation target is high enough to drive the real interest rate to its negative natural level using standard monetary policy tools, so the central bank can set a positive nominal rate and a low interest rate does not appear. This case is depicted by the full employment (FE) equilibrium A in Figure 3. As the AD kink lies to the right of AS_{HT}, AD_{HT} crosses the AS curve in its vertical segment, and the resulting equilibrium has output at the potential level and inflation at the target.

The condition (31) in contrast does not hold for $\Pi_{kink} > \overline{\Pi}$, because a low inflation target prevents the central bank from pushing the real rate down to a level that would

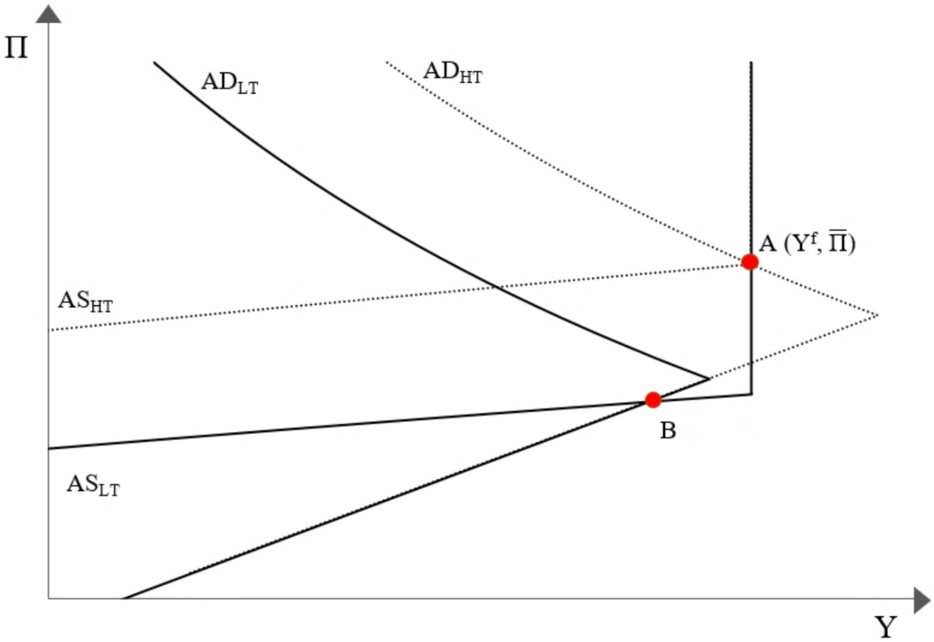

Figure 3: FE and LIR Equilibria in a Bubbleless Economy

be consistent with potential output. As the real interest rate is above the natural level despite a binding ZLB, inflation falls below the target, and downwardly rigid wages prevent market clearing in the labour and goods markets ($Y < Y^f$). The resulting low interest rate (LIR) equilibrium corresponds to point B, where the demand curve AD_{LT} intersects the supply curve AS_{LT} in its upward sloping segment (Figure 3).[15]

The comparison between these two equilibria explains perfectly the effect of a higher inflation target when the natural interest rate turns negative. Raising the inflation target gives the central bank more room to cut the nominal interest rate, preventing a ZLB episode.

3.2 A Bubbly Economy

The steady state value of the price of unleveraged and leveraged bubbles can be zero or positive. When it is positive, the economy is bubbly. I first study the condition under which *fully* unleveraged bubbles arise in FE and LIR equilibria at a negative natural interest rate, and their impact when the bubbleless economy experiences a low interest rate. Then I do the same for *fully* leveraged bubbles.

[15] The LIR equilibrium is determinate and unique like the "secular stagnation" equilibrium in Eggertsson and Mehrotra (2014), while the uniqueness of an FE steady state such as A stands in contrast with the result of Eggertsson et al. (2017). I find a unique equilibrium featuring FE, when $\frac{1}{\overline{\Pi}} \leq (1 + r^f) < 1$, because the minimum nominal wage level is indexed to the inflation target in equation (8). This induces agents to adjust inflation expectations upward when the central bank raises the inflation target.

3.2.1 Unleveraged Bubbles

If $\phi = 0$, borrowers optimally choose not to hold bubbly assets, so the bubble is fully unleveraged ($b_t^B = 0$ and $b_t^L = \frac{1}{\chi}$) and there is no default ($\gamma_{t+1} = h_{t+1} = 0$). The bubble dynamics is expressed by the equation (22), which becomes:

$$p_{t+1}^b = \left(\frac{1}{1-\rho}\right)(1+r_t)\,p_t^b$$

by imposing $\lambda_{b,t}^L = 0$. There are two steady state values of p^b that satisfy this equation, and they can be derived by substituting for $(1+r)$ and solving the resulting polynomial. The first root of the polynomial corresponds to the bubbleless case ($p^b = 0$), while the second one, denoted to the superscript ub, is associated with an unleveraged bubbly steady state:

$$p^{b,ub} = \chi\frac{\beta}{1+\beta}Y^L - \frac{1}{(1-\rho)}(1-\chi)\,D$$

which exists ($p^{b,ub} > 0$) if:

$$\frac{(1-\chi)}{\chi}\left[\frac{(1+\beta)\,D}{\beta Y^L}\right] = (1+r) < (1-\rho) \tag{32}$$

Equation (32) is the standard condition for the existence of stochastic rational bubbles (e.g., Weil, 1987; Bengui and Phan, 2016). If there is not a sufficient store of value in the bubbleless economy, inherently worthless assets guarantee a higher return than bonds, as long as the probability that they will be valued in the future is sufficiently high.

If the natural interest rate is negative, $(1+r) = (1+r^f)$ in an FE steady state, because condition (31) holds; equation (31) does not hold in an LIR equilibrium and $(1+r^f) < (1+r) = \frac{1}{\Pi}$. Therefore an unleveraged bubble is more likely to emerge in an FE equilibrium, and its size p^b, which is the gap between saving and borrowing in a bubbleless economy, will be larger. This result derives from the higher inflation target that is typical of the FE equilibrium and drives the real interest rate to extremely negative values. The real interest rate is negative in an LIR steady state too, because a negative natural rate forces the central bank to hit the ZLB. However, the real rate in this case can be pushed less deep into negative territory ($1 < \Pi < \overline{\Pi}$), and this makes an unleveraged bubbly episode less likely to occur, and smaller if it does. In any case, a negative natural rate of interest is not a sufficient condition for unleveraged bubbles with either FE or LIR. Specifically, the existence of an unleveraged bubble depends on the inflation level in an LIR equilibrium, among other factors.

Next I investigate the effect of a fully unleveraged bubble that emerges in an LIR equilibrium. With an unleveraged bubble, the transitional dynamics of the economy is straightforward, as for $p_0^b < p^b$, the economy gradually converges to $p^b = 0$, while for $p_0^b > p^b$, the bubble price follows an explosive path and no equilibrium exists; finally, if $p_0^b = p^b$, the economy lies at the bubbly equilibrium. This means that the economy immediately reaches the fully unleveraged bubbly equilibrium, once the price of the bubble is at the positive steady state value. This equilibrium is an asymptotic bubbly one ($\lim_{t\to\infty} p_t^b > 0$).

Unleveraged bubbly assets do not affect the demand for borrowing (23), but they alter the supply of credit from lenders, which is:

$$
\begin{aligned}
S^{c,ub} &= \frac{\beta}{1+\beta}Y^L - \frac{\beta}{1+\beta}p^b b^L - \frac{1}{1+\beta}p^b b^L \\
&= \frac{\beta}{1+\beta}Y^L - p^b b^L
\end{aligned}
$$

In contrast to their behaviour in a bubbleless economy, lenders reallocate their savings by investing the amount $\frac{\beta}{1+\beta}p^b b^L$ in bubbly assets. They also reduce the amount of credit by $\frac{1}{1+\beta}p^b b^L$, because bubbles provide additional income in old age, inducing them to save less.[16] As the credit supply is lower, the equilibrium real interest rate is higher than in a bubbleless economy:

$$
\left(1+r^{ub}\right) = \frac{(1-\chi)}{\chi}\left[\frac{(1+\beta)\,D}{\beta Y^L - (1+\beta)\,p^b b^L}\right] \tag{33}
$$

Bubbles redistribute resources from young borrowers, who can raise less in funds because of the lower supply of credit, to old lenders, who get a higher income from their investment in bubbly assets; and the resulting allocation corresponds to a higher equilibrium real interest rate and so to a higher natural rate of interest.

If the bubbleless economy lies at the LIR equilibrium B in Figure 4, and then unleveraged bubbly assets appear, the bubble pushes the natural interest rate up and so Π_{kink} is reduced. This is shown graphically by the aggregate demand curve moving towards AD_0, because the AD kink shifts down (Figure 4).

As bubbly assets drive the natural interest rate up to $(1-\rho)$, closing the gap between supply and demand for credit,[17] the economy no longer experiences an LIR, if:

$$
1-\rho \geq \frac{1}{\overline{\Pi}}
$$

which corresponds to $\Pi_{kink} \leq \overline{\Pi}$. As $\overline{\Pi} > \Pi = \frac{1}{1+r}$ in an LIR equilibrium, the probability of the bubble surviving is necessarily higher than the inverse of the inflation target. So if unleveraged bubbles arise, the economy reaches an FE equilibrium (A in Figure 4).

[16] I provide a formal derivation of the credit supply in Appendix B. There, I also compute aggregate demand in fully unleveraged and leveraged bubbly steady states, and I study the redistribution of resources caused by the two bubble types.

[17] This is a standard result in the literature on rational bubbles (e.g., Samuelson, 1958; Tirole, 1985), where asset price bubbles emerge in a dynamically inefficient economy to restore efficiency.

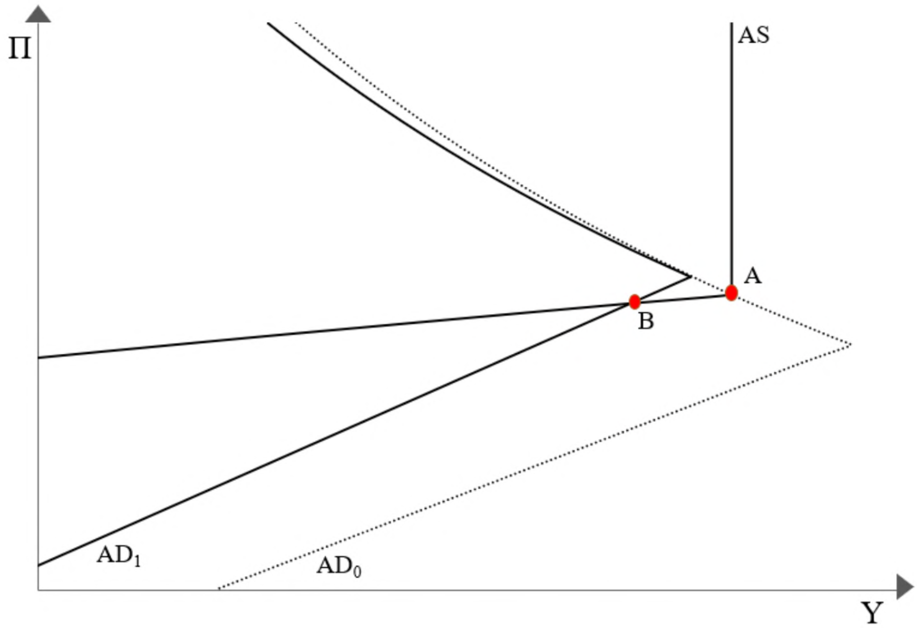

Figure 4: FE and LIR Equilibria in a Bubbly Economy

3.2.2 Leveraged Bubbles

In the opposite case of $\phi = 1$, bubbles are fully leveraged ($b^B = \frac{1}{1-\chi}$ and $b^L = 0$).[18] The law of motion of the bubble is derived from (16) at $\lambda^B_{b,t} = 0$:

$$p^b_{t+1} = (1 + r_t)\, p^b_t$$

Like with unleveraged bubbles, there are two steady state values that satisfy the equation describing the bubble dynamics; one is zero, the other is:

$$p^{b,lb} = \chi \frac{\beta}{1+\beta} Y^L - (1-\chi)\, D$$

where the superscript lb denotes a leveraged bubbly equilibrium. The price of a leveraged bubbly asset is positive if:

$$\frac{(1-\chi)}{\chi} \left[\frac{(1+\beta)\, D}{\beta Y^L} \right] = (1+r) < 1 \tag{34}$$

From an economic point of view, equation (34) does not substantially differ from (32). Rational bubbles can only appear if the bubbleless economy lacks sufficient opportunities for investment and so the supply of saving exceeds the demand for borrowing. Unlike in

[18] I impose:

$$D < \frac{T}{1+\beta} - \frac{\beta}{1+\beta} \left[(1+r_t) \left(Y^B_t - T - p^b_t b^B_t \right) + p^b_{t+1} b^B_t \right]$$

which guarantees a binding borrowing limit when bubbly assets can be fully collateralised.

(32) however, the probability of the bubble bursting does not enter equation (34), and this derives from the *risk-shifting* that is typical of leveraged bubbly episodes (Bengui and Phan, 2016). When bubbles can be fully collateralised, their expected discount return is zero, but borrowers purchase them because the collateral value is higher than the bubble price if the real interest rate is negative (see equation (16)). As borrowers do not invest with their own money, they do not internalise the risk of a bubble collapse, and bubbly assets no longer need to have a sufficiently high probability of surviving in order to have a valuation. Borrowers also overvalue bubbly assets, making the bubble larger than the unleveraged bubble ($p^{b,lb} > p^{b,ub}$). Therefore leveraged bubbles are riskier and larger than unleveraged ones.

This fundamental distinction between the two bubble types is emphasised when there is a negative natural interest rate. Unlike in an unleveraged bubble, this is a sufficient condition for leveraged bubbly episodes in both the FE and LIR steady states, because a negative natural rate induces the central bank to push the real interest rate below zero. Furthermore, as the real interest rate is higher than its natural level with an LIR, the gap between saving and borrowing is larger at the full employment level of output, and the same applies to the size of a leveraged bubble.

Next I study how leveraged bubbles affect economic outcomes when the bubbleless economy experiences an LIR. The transitional dynamics is the same as above. The supply of credit is the same as in a bubbleless economy (equation (24)), while the demand for borrowing becomes:

$$D^{c,lb} = \frac{(1-\chi)}{\chi}\left(\frac{D + p^b b^B}{1+r}\right)$$

This equation is obtained from the definition of credit demand by using $d^B = \left(\frac{D+p^b b^B}{1+r}\right)$. Purchasing bubbles allows borrowers to demand more funds than in a bubbleless economy, and a higher equilibrium real interest rate directly follows from the increased credit demand:

$$\left(1+r^{lb}\right) = \frac{(1-\chi)}{\chi}\left[\frac{(1+\beta)\left(D+p^b b^B\right)}{\beta Y^L}\right] \tag{35}$$

Higher real and natural interest rates reflect the redistribution that leveraged bubbles cause, channelling resources from young borrowers to old lenders like in unleveraged bubbles, but through different channels. Although young borrowers demand more funds by pledging bubbly assets, the supply is fixed as is the consumption of young lenders. This means that borrowers take on more debt, but they just pay higher interest rates without collecting extra funds and without gaining compensation from the bubble purchases. In old age, the borrowers find their proceeds from bubbles are exhausted by the higher debt, while the proceeds of lenders from bonds increase.

The effect a fully leveraged bubble has on an economy that lies at an LIR equilibrium is the same as that of a fully unleveraged one. The natural rate of interest goes up and Π_{kink} goes down, as depicted by the downward shift of the AD kink, which moves the AD curve from the original position AD_1 towards AD_0 (Figure 4). As a leveraged bubbly episode absorbs extra savings, leading to a non-negative natural interest rate, equation (31) holds ($\overline{\Pi} > 1$) and the economy goes from equilibrium B to A in Figure 4.

Although both unleveraged and leveraged bubbles allow the central bank to escape from a ZLB episode, the two bubble types emerge under different conditions in an LIR equilibrium and have a differential effect on the natural rate of interest. Unleveraged bubbles occur only with a negative natural interest rate and a sufficiently high inflation rate and when there is a sufficiently low probability of them bursting. Leveraged bubbles occur whenever the natural rate turns negative and they raise it more ($1 > 1 - \rho$).

4 Conclusions

The historical decline in the natural interest rate in advanced economies has caused low risk-free interest rates in the decade 2008–2018. I investigate the interplay between two effects of persistently low interest rates, which are the increased probability of ZLB episodes and the heightened risk of *rational* asset price bubbles. Specifically, I have studied how the bubble type shapes the interplay between these two effects using an OLG model that features income inequality, downward nominal wage rigidity and rational bubbles.

Asset price bubbles can be leveraged or unleveraged, and this makes a substantial difference. Leveraged bubbles naturally emerge with a negative natural interest rate and have a higher probability of bursting, but they raise the natural rate of interest a long way, giving the central bank more space to manoeuvre the policy rate. Unleveraged bubbles have a higher probability of surviving, but they do not necessarily appear if the natural interest rate is negative. Furthermore, they raise the natural rate in a limited manner and this only partially mitigates the risk of hitting the ZLB. Hence a negative natural rate of interest not only generally fosters rational asset price bubbles, but it also provides fertile ground for leveraged ones above all. As this bubbly type is also the most effective at raising the natural rate, asset price bubbles give the central bank more space to cut the nominal interest rate at the cost of a higher probability of bursting.

The existence of bubbly episodes does not only depend on there being a negative natural rate of interest, which forces the central bank to push the real rate below zero using the standard monetary policy tools, as it is also affected by the inflation target. If the targeted level of inflation is high, the central bank can accommodate a negative natural rate without hitting the ZLB, but this pushes the real interest rate down further, fostering all bubble types.

Although the present work sheds light on the different sources and impacts of leveraged and unleveraged bubbly episodes in a low interest rates environment, the analysis is limited to bubbly steady states in which bubbles do not burst. This does not allow the different consequences in terms of financial stability of unleveraged and leveraged bubbles to be captured, but this could be analysed by adding a banking sector to the model and considering transitory bubbly episodes. Furthermore, I restrict the analysis to fully unleveraged and leveraged bubbles that can emerge for a constant pledgeability of the bubbly assets. If the parameter ϕ is allowed to vary and it is manoeuvred by a macroprudential authority, the optimal behaviour for a policy maker facing different transitory bubbly episodes could be studied.

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Appendix

A Fully Unleveraged and Leveraged Bubbly Equilibria: Bubble Holding

A.1 A Fully Unleveraged Bubbly Equilibrium

If $\phi = 0$, borrowers always repay their debt and there is no default ($\gamma = h = 0$). The no-arbitrage equation (22) takes the following form by imposing $\lambda_b^L = 0$:

$$(1+r) = (1-\rho)$$

Lenders hold bubbles in a fully unleveraged bubbly steady state only if the real interest rate is equal to the probability of the bubble surviving. The first order conditions for the borrowers taken at this steady state are:

$$\lambda_b^B = \left[\frac{1}{C_y^B} - \beta(1-\rho)\frac{1}{C_o^B}\right] p^b$$

$$\lambda_d^B = \frac{1}{(1+r)}\frac{1}{C_y^B} - \beta\frac{1}{C_o^B}$$

Given that the borrowing constraint is binding in a bubbleless economy ($\lambda_d^B > 0$):

$$\frac{1}{(1+r)\,C_y^B} > \beta\frac{1}{C_o^B}$$

Therefore, if $(1+r) = (1-\rho)$:

$$\lambda_b^B = \left[\frac{1}{C_y^B} - \beta(1+r)\frac{1}{C_o^B}\right] p^b > 0$$

This means borrowers do not demand bubbles in a fully unleveraged bubbly steady state.

A.2 A Fully Leveraged Bubbly Equilibrium

If $\phi = 1$ and $\gamma = 1$, $h = \frac{\phi p^b b^L}{D+\phi p^b b^L} = \frac{p^b b^L}{D+p^b b^L} < 1$. Equation (16) in this case becomes:

$$\lambda_b^B = \frac{1}{C_y^B}\left[1 - \frac{1}{(1+r)}\right] p^b$$

Borrowers demand bubbly assets if $(1+r) = 1$. The no-arbitrage condition (22) of lenders reduces to:

$$(1-h\rho) = (1-\rho)$$

by setting $\lambda_b^L = 0$. As $h < 1$, the marginal cost of investing in bubbly assets on the left-hand side of the equation is higher than the marginal benefit on the right-hand side. So lenders do not invest in bubbly assets ($\lambda_b^L > 0$).

B Fully Unleveraged and Leveraged Bubbly Equilibria: Distribution of Resources and Aggregate Demand

B.1 A Fully Unleveraged Bubbly Equilibrium

In a fully unleveraged bubbly equilibrium, equations (11) and (12) reduce to:

$$C_{y,t}^{B} = Y_t^B + d_t^B - T$$

$$C_{o,t+1}^{B} = T - (1 + r_t)\, d_t^B$$

Equations (17) and (18) become:

$$C_{y,t}^{L} = Y_t^L - d_t^L - p_t^b b_t^L$$

$$C_{o,t+1}^{L} = \theta_{t+1} p_{t+1}^b b_t^L + (1 + r_t)\, d_t^L$$

By combining the last two equations with (21), we get the credit supply:

$$S_t^{c,ub} = \frac{\beta}{1+\beta}\left(Y_t^L - p_t^b b_t^L\right) - \frac{(1-\rho)\, p_{t+1}^b b_t^L}{(1+\beta)\,(1+r_t)}$$

which becomes at the steady state:

$$S^{c,ub} = \frac{\beta}{1+\beta} Y^L - \frac{\beta}{1+\beta} p^b b^L - \frac{1}{1+\beta} p^b b^L$$

by using the equation of the bubble dynamics. Unleveraged bubbles not only change the supply of credit, but they generally redistribute resources as shown by the steady state values of the other main variables:

$$\begin{aligned}
d^{B,ub} &= \frac{D}{(1+r)} = \frac{\chi}{(1-\chi)}\left(\frac{\beta}{1+\beta} Y^L - p^b b^L\right) \\
d^{L,ub} &= S^{c,ub} = \frac{(1-\chi)}{\chi}\frac{D}{(1+r)} = \frac{\beta}{1+\beta} Y^L - p^b b^L \\
C_y^{B,ub} &= Y^L + d^{B,ub} - T \\
C_o^{B,ub} &= T - (1+r) d^{B,ub} = T - D \\
C_y^{L,ub} &= Y^L - d^{L,ub} - p^b b^L = \frac{1}{1+\beta} Y^L \\
C_o^{L,ub} &= p^b b^L + (1+r) d^{L,ub} = p^b b^L + \frac{(1-\chi)}{\chi} D
\end{aligned}$$

As mentioned above, young borrowers consume less and old lenders consume more because of unleveraged bubbly assets. The consumption of old borrowers and young lenders is the same as that in the bubbleless equilibrium. Though fewer funds are available for borrowers in the credit market, a higher interest rate is charged and so the debt repaid by the old borrowers (D) is unchanged. Furthermore, bubbly assets induce young lenders to save less, but these extra resources are fully exhausted by the bubble purchases. The redistribution of resources caused by an unleveraged bubble is also reflected in a different aggregate demand, which is:

$$Y_{AD}^{ub} = (1-\chi)\, Y^B + \chi \left(\frac{1+\beta}{\beta}\right) p^b b^L + (1-\chi) \left(\frac{1+\beta}{\beta}\right) \frac{\kappa}{\Pi^{\phi_\Pi - 1}} D$$

for a positive nominal interest rate, while it becomes:

$$Y_{AD}^{ub} = (1-\chi)\, Y^B + \chi \left(\frac{1+\beta}{\beta}\right) p^b b^L + (1-\chi) \left(\frac{1+\beta}{\beta}\right) \Pi D$$

when the policy rate is zero. Both equations are computed from (9), (10) and (33).

B.2 A Fully Leveraged Bubbly Equilibrium

In a fully leveraged bubbly equilibrium, the budget constraints of the borrowers are:

$$C_{y,t}^B = Y_t^B + d_t^B - p_t^b b_t^B - T$$

$$C_{o,t+1}^B = T + \theta_{t+1} p_{t+1}^b b_t^B - (1-\gamma_{t+1})(1+r_t)\, d_t^B - \gamma_{t+1}\left(D + \phi \theta_{t+1} p_{t+1}^b b_t^B\right)$$

while the budget constraints of the lenders (17) and (18) collapse to:

$$C_{y,t}^L = Y_t^L - d_t^L$$

$$C_{o,t+1}^L = (1-h_{t+1})(1+r_t)\, d_t^L$$

By considering the steady state value of the main variables, we obtain:

$$d^{B,lb} = \left[\frac{D + p^b b^B}{(1+r)}\right] = \frac{\chi}{(1-\chi)} \frac{\beta}{1+\beta} Y^L$$

$$d^{L,lb} = S^{c,lb} = \frac{(1-\chi)}{\chi}\left[\frac{D + p^b b^B}{(1+r)}\right] = \frac{\beta}{1+\beta} Y^L$$

$$C_y^{B,lb} = Y^L + d^{B,lb} - p^b b^B - T$$

$$C_o^{B,lb} = T + p^b b^B - (1+r) d^{B,lb} = T - D$$

$$C_y^{L,lb} = Y^L - d^{L,lb} = \frac{1}{1+\beta} Y^L$$

$$C_o^{L,lb} = (1+r) d^{L,lb} = \frac{(1-\chi)}{\chi}\left(D + p^b b^L\right)$$

because leveraged bubbles do not burst ($h = \gamma = 0$). The reallocation of resources caused by a fully leveraged bubble translates into a different aggregate demand from that of the bubbleless economy. The new aggregate demand is still characterised by two regimes. It is:

$$Y_{AD}^{lb} = (1-\chi)\, Y^B + (1-\chi)\left(\frac{1+\beta}{\beta}\right) \frac{\kappa}{\Pi^{\phi_\Pi - 1}} \left(D + p^b b^B\right)$$

when the ZLB is not binding. It becomes:

$$Y_{AD}^{lb} = (1-\chi)\, Y^B + (1-\chi)\left(\frac{1+\beta}{\beta}\right) \Pi \left(D + p^b b^B\right)$$

when the central bank hits the ZLB. These equations are derived from equations (9), (10) and (35).

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